On the Limiting Ratio of Current Age to Total Life for Null Recurrent Renewal Processes

Blanchet, J., Glynn, P. and Thorisson, H.

March 31, 2015

Abstract

If the inter-arrival time distribution of a renewal process is regularly varying with index $\alpha \in (0,1)$ (i.e. the inter-arrival times have infinite mean) and if $A(t)$ is the associated age process at time t. Then we show that if $C(t)$ is the length of the current cycle at time t,

$$A(t)/C(t) \Rightarrow U^{1/\alpha},$$

where U is $U(0,1)$. This extends a classical result in renewal theory in the finite mean case which indicates that the limit is $U(0,1)$.

1 The Result

In this note we revisit some classical renewal theorems for infinite mean inter-arrival time distributions. Consider a sequence of i.i.d. non-arithmetic non-negative random variables $\{X_n : n \geq 1\}$. Set $S_0 = 0$ and define $S_n = X_1 + ... + X_n$ for $n \geq 1$. Further, consider the associated renewal process

$$N(t) = \max\{n \geq 0 : S_n \leq t\},$$

and the corresponding age, residual life-time, and cycle-in-progress processes defined as

$$A(t) := t - S_{N(t)}, \quad B(t) = S_{N(t)+1} - t, \quad C(t) = B(t) + A(t).$$

It is well known that if $EX_n < \infty$, then $(A(t), C(t)) \Rightarrow (CU, C)$, where

$$P(C \in A) = E(X_1 I(X_1 \in A))/E(X_1),$$

and $U \sim U(0,1)$, see for example, Asmussen (2003).

Our goal here is to investigate the case in which $EX_n = \infty$. In particular, we assume that X_n's have a regularly varying density at infinity with index $\alpha + 1$ and $\alpha \in (0,1)$; that is, assume that there exists $t_0 > 0$ such that for all $t > t_0$

$$\bar{F}(t) := P(X_n > t) = \int_t^\infty s^{-\alpha-1} L(s)\, ds,$$

for a slowly varying function $L(\cdot)$. We will prove the following theorem.

Theorem 1.

$$A(t)/C(t) \Longrightarrow U^{1/\alpha}$$

as $t \to \infty$.

Proof. First we obtain a renewal equation for the distribution of the regenerative process $V(t) = A(t)/C(t)$, namely

$$\begin{aligned} a(t) &= P(V(t) > x) \\ &= \int_t^\infty P(V(t) > x | \tau_1 = s) P(\tau_1 \in ds) + \int_0^t a(t-s) P(\tau_1 \in ds) \\ &= \int_t^\infty I(t/s > x) P(\tau_1 \in ds) + \int_0^t a(t-s) P(\tau_1 \in ds) \\ &= \int_t^{t/x} P(\tau_1 \in ds) + \int_0^t a(t-s) P(\tau_1 \in ds) \\ &= b(t) + \int_0^t a(t-s) F(ds), \end{aligned}$$

where $b(t) = \bar{F}(t) - \bar{F}(t/x)$. We then conclude that

$$a(t) = \int_0^t b(t-s) u(ds),$$

with $u(s) = E(N(s) + 1)$ being the renewal function. We have from Theorem 5 of Erickson (1970) that

$$u(t) \sim c^*/\bar{F}(t). \tag{1}$$

(This property actually holds true even if $\alpha \in [0,1]$.) We then need to evaluate the limit of

$$a(t) = \int_0^1 \bar{F}(t(1-r)) u(tdr) - \int_0^1 \bar{F}(t(1-r)/x) u(tdr)$$

as $t \to \infty$. We shall argue the (quite intuitive, due to (1)) limits,

$$\int_0^1 \bar{F}(t(1-r)) u(tdr) \sim c^* \int_0^1 \frac{\bar{F}(t(1-r))}{\bar{F}(t)} \cdot \frac{u(tdr)}{u(t)} \sim c^*\alpha \int_0^1 (1-r)^{-\alpha} r^{\alpha-1} dr, \tag{2}$$

and similarly

$$\int_0^1 \bar{F}(t(1-r)/x) u(tdr) \sim c^*\alpha x^\alpha \int_0^1 (1-r)^{-\alpha} r^{\alpha-1} dr, \tag{3}$$

thereby concluding that

$$P(V(t) > x) \sim c^*\alpha(1 - x^\alpha).$$

By tightness have that $c^*\alpha = 1$ and hence, provided that (2) and (3) we will be able to conclude that

$$P(V(t) > x) \sim (1 - x^\alpha) = P\left(U^{1/\alpha} > x\right)$$

as $t \to \infty$. Theorem 2 in Teugels (1968) actually indicates that the asymptotics in (2) and (3) are indeed correct, and for this part it is important to assume that $\alpha \in (0,1)$ and that the slowly varying part component of $\bar{F}(\cdot)$ satisfies some mild conditions (which are satisfied precisely if X_n has a slowly varying density as we have assumed). $\square$

Remark 1: It is desirable to show the result for $\alpha \in [0, 1]$ and we would like to remove the conditions on the slowly varying assumption on $\bar{F}(\cdot)$ in Teugels (1968). There are results obtained for

Remark 2: We finish this note with a comment. It turns out that the joint distribution of $(A(t), B(t))/t$ as $t \to \infty$ was derived by Dynkin (1955) and Lamperti (1962), (see, for example, Theorem 8.6.3 in Bingham, Goldie and Teugels (1987)). In principle, the law obtained above might be derived from the Dynkin-Lamperti theorem, but such derivation does not appear to be as direct as our derivation above. Maybe this explains why the simple asymptotic limit obtained for $A(t)/C(t)$ as $t \to \infty$ appears to not have been explicitly identified in well known references for the Dynkin-Lamperti theorem.

Acknowledgement 1. *This research was partially supported by the grants DMS-0806145, CMMI-0846816 and CMMI-1069064. This result was presented as a solution to an open problem discussed during the Stochastic Networks Conference organized at the Newton Institute in 2013.*

References

[Asmussen (2003)] Asmussen, S. (2003). *Applied Probability and Queues* (2nd ed.). Springer-Verlag.

[Bingham et al. (1987)] Bingham, N., Goldie, C., and Teugels, J. (1987). *Regular Variation,* Cambridge University Press, Cambridge.

[Dynkin (1955)] Dynkin, E. B. (1955). Limit theorems for sums of independent random quantities. *Izves. Akad. Nauk U.S.S.R.*, 19, 247-266.

[Erickson (1970)] Erickson, K. B. (1970). Strong renewal theorems with infinite mean. *Transactions of the American Mathematical Society*, 151, 263-291.

[Lamperti (1962)] Lamperti, J. (1962). An invariance principle in renewal theory. *Annals of Mathematical Statistics*, 33, 685-696.

[Teugels (1968)] Teugels, J. (1968). Renewal Theorems When the First or the Second Moment is Infinite. *Ann. Math. Statist.*, 39, 1210–1219.